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CONFIDENCE BOUNDS FOR MAGNITUDE-SQUARED COHERENCE ESTIMATES.(U)
JUL 78 G C CARTER, E H SCANNELL
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Confidence Bounds for Magnitude-Squared Coherence Estimates

A Paper Presented at the
1978 IEEE International Conference on
Acoustics, Speech, and Signal Processing.

G. Clifford Carter
Everett H. Scannell, Jr.
Special Projects Department

13 July 1978

NUSC
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PREFACE

This document was prepared under NUSC Project No. A-126-10, "Interarray Processing," Principal Investigators, J. B. Hall, Jr. (Code 3211), and G. C. Carter (Code 313); Program Manager, R. Cockerill (NAVSEA-06H2-51), Naval Sea Systems Command.

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Magnitude Squared Coherence	TMA

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This document presents both the oral and written versions of a paper presented (in 15 minutes) on 12 April 1978 at the 1978 IEEE International Conference on Acoustics, Speech, and Signal Processing, in Tulsa, Oklahoma.

The main emphasis of the talk was on explaining coherence and its usefulness. The paper given in the coherence record emphasizes how to estimate coherence and how accurately this can be done. In underwater acoustics where next page

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cont.

signals are digitally processed at the outputs of two or more receiving sensors, it is desirable to estimate the coherence spectrum, both for detection and position estimation.

A processing technique for computing arbitrary confidence bounds for stationary Gaussian signals is presented. New computationally difficult examples are given for 80-95 percent confidence with independent averages of 8, 16, 32, 64, and 128. A discussion of the computational difficulties together with algorithmic details (including the FORTRAN program) are presented.

CONFIDENCE BOUNDS FOR MAGNITUDE-SQUARED COHERENCE ESTIMATES

- ***What is coherence?***
- ***How and how accurately do you estimate it?***

THE PURPOSE OF THIS TALK IS TO ANSWER TWO FUNDAMENTAL QUESTIONS: FIRST, WHAT IS COHERENCE; SECOND, HOW DO YOU ESTIMATE COHERENCE AND HOW ACCURATE CAN THIS ESTIMATION BE.

THE MAIN EMPHASIS OF THIS TALK IS THE EXPLANATION OF COHERENCE AND ITS USEFULNESS. THE PAPER GIVEN IN THE CONFERENCE RECORD EMPHASIZES HOW TO ESTIMATE COHERENCE AND HOW ACCURATELY THIS CAN BE DONE. THE IMPORTANCE OF DETERMINING CONFIDENCE BOUNDS FOR ESTIMATES OF COHERENCE WILL ONLY BE APPARENT TO SOMEONE WHO WANTS TO ESTIMATE COHERENCE. THUS, THE TALK THIS MORNING WILL SHOW HOW USEFUL THE COHERENCE IS AND HOW TO USE THE RESULTS IN THE COHERENCE RECORD TO DETERMINE THE ACCURACY WITH WHICH THE COHERENCE CAN BE ESTIMATED.

-NEXT SLIDE PLEASE-

$$\gamma_{ab}(f) = \frac{G_{ab}(f)}{\left[G_a(f)\, G_b(f)\right]^{1/2}}$$

$$0 \le \left|\gamma_{ab}(f)\right|^2 \le 1, \ \forall f$$

a,b either source, receiver pair or receiver, receiver pair

THE TERM COHERENCE HAS SEVERAL DIFFERENT MEANINGS AND DEFINITIONS. THE ONE WE USE HERE IS THE COMPLEX COHERENCE OR COEFFICIENT OF COHERENCY DEFINED BY WEINER IN 1930. FOR OUR PURPOSES, WE DEFINE THE COHERENCE BETWEEN TWO STATIONARY RANDOM PROCESSES, A AND B, AS THE CROSS POWER SPECTRUM DIVIDED BY THE SQUARE ROOT OF THE PRODUCT OF THE AUTO POWER SPECTRA. THE COHERENCE IS A FUNCTION OF FREQUENCY AND HAS THE USEFUL PROPERTY THAT ITS MAGNITUDE SQUARED IS BOUNDED BETWEEN ZERO AND UNITY. IT IS A NORMALIZED CROSS SPECTRAL DENSITY THAT, IN SOME SENSE, MEASURES THE EXTENT TO WHICH TWO RANDOM PROCESSES ARE SIMILAR. FOR EXAMPLE, TWO UNCORRELATED RANDOM PROCESSES ARE INCOHERENT; THAT IS, THE COHERENCE IS ZERO BETWEEN UNCORRELATED PROCESSES. FURTHER, THE COHERENCE BETWEEN TWO LINEARLY RELATED PROCESSES IS UNITY. THE TWO PROCESSES UNDER CONSIDERATION CAN BE AN UNDERWATER ACOUSTIC SOURCE AND RECEIVER PAIR OR TWO RECEIVER PAIRS.

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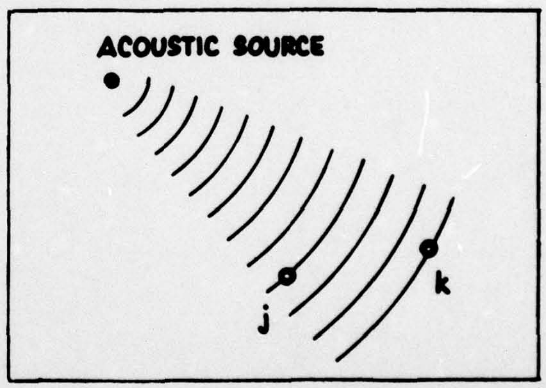

ONE PHYSICAL PROBLEM THAT MOTIVATES THIS RESEARCH IS THE DESIRE TO PASSIVELY ESTIMATE GEOGRAPHICAL INFORMATION ABOUT THE STATE OF AN ACOUSTIC SOURCE. IN THE DEVELOPMENT HERE, AN ACOUSTIC POINT SOURCE RADIATES SPHERICAL WAVES THAT ARE RECEIVED FIRST AT ONE SENSOR AND SOME DELAYED TIME LATER AT A SECOND SENSOR. THE SOURCE IS ASSUMED STATIONARY FOR THE OBSERVATION PERIOD AND THE SENSOR SEPARATION IS ASSUMED KNOWN. EACH RECEIVED WAVEFORM IS OBSERVED IN THE PRESENCE OF UNCORRELATED NOISE. THE PROBLEM WE ADDRESS HERE IS THE PHYSICAL INTERPRETATION OF THE COHERENCE FOR THIS MODEL.

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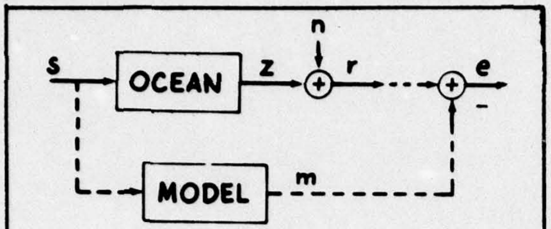

$$|\gamma_{sr}|^2 = \frac{G_m}{G_r}$$

$$\frac{G_z}{G_n} = \frac{|\gamma_{sr}|^2}{1 - |\gamma_{sr}|^2}$$

A SOURCE SIGNAL S EXCITES THE MEDIUM TO YIELD AN OUTPUT Z. THIS OUTPUT Z IS CORRUPTED BY ADDITIVE NOISE n AND RECEIVED AS r. WE CONSTRUCT A LINEAR MODEL OF THE MEDIUM THAT GENERATES AN OUTPUT m. BY PROPER CHOICE OF THE MODEL WE CAN MINIMIZE THE MEAN SQUARE ERROR e, OR DIFFERENCE BETWEEN THE RECEIVED SIGNAL r AND MODEL OUTPUT m. THE MAGNITUDE SQUARED COHERENCE BETWEEN SOURCE AND RECEIVER IS GIVEN BY THE RATIO OF THE MODEL OUTPUT POWER TO THE RECEIVER OUTPUT POWER. SINCE GAMMA SQUARED IS BOUNDED BY UNITY, IT PROVIDES AN INDICATION OF WHAT PORTION OF THE RECEIVED POWER CAN BE ATTRIBUTED TO A MINIMUM MEAN SQUARE ERROR LINEAR MODEL OF THE OCEAN MEDIUM. THE POWER RATIO OF THE OCEAN OUTPUT DUE TO THE SOURCE VERSUS AMBIENT IS ALSO DIRECTLY RELATED TO THE SOURCE-TO-RECEIVER COHERENCE. IN PARTICULAR, THIS SIGNAL-TO-NOISE RATIO IS GIVEN BY GAMMA SQUARED OVER ONE MINUS GAMMA SQUARED.

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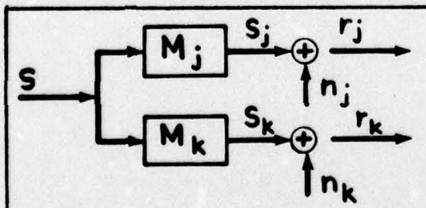

$$|\gamma_{r_j r_k}|^2 = |\gamma_{sr_j}|^2 \, |\gamma_{sr_k}|^2$$

$$\frac{G_s}{G_n} = \frac{|\gamma_{r_j r_k}|}{1 - |\gamma_{r_j r_k}|}$$

IN THE GENERAL CASE, WE CAN MODEL THE ACOUSTIC PROPAGATION OF A SINGLE ACOUSTIC SOURCE AND NOISE CORRUPTED RECEPTION AT TWO RECEIVERS AS SHOWN HERE. IN PARTICULAR, WE TREAT THE PATH FROM THE SOURCE TO EACH RECEIVER AS A LINEAR TIME INVARIANT FILTER. THE RECEIVER SIGNALS r SUB J AND r SUB K CONSIST OF THE FILTER OUTPUTS PLUS NOISE.

A SPECIAL CASE OF THIS MODEL IS WHEN THE FIRST RECEIVER WAVEFORM CONSISTS OF SIGNAL PLUS NOISE, AND THE SECOND RECEIVED WAVEFORM CONSISTS OF AN ATTENUATED AND DELAYED SIGNAL IN THE PRESENCE OF UNCORRELATED NOISE. THE MATHEMATICAL PROBLEM OF ESTIMATING THE TIME DELAY OR EQUIVALENT SOURCE BEARING AND, THUS, SOURCE RANGE, IS CLOSELY RELATED TO COHERENCE.

UNDER CERTAIN ASSUMPTIONS WE CAN SHOW THAT THE MAGNITUDE SQUARED COHERENCE BETWEEN TWO RECEIVER PAIRS IS THE PRODUCT OF THE INDIVIDUAL SOURCE-TO-RECEIVER COMBINATIONS. THUS, THE RECEIVED SIGNAL-TO-NOISE RATIO IS THE RECEIVER-TO-RECEIVER MAGNITUDE COHERENCE OVER ONE MINUS THE RECEIVER-TO-RECEIVER MAGNITUDE COHERENCE.

-NEXT SLIDE PLEASE-

$$\hat{\gamma}_{ab} = \frac{\sum_{n=1}^{N} A_n B_n^*}{\left[\sum_{n=1}^{N} |A_n|^2 \sum_{n=1}^{N} |B_n|^2\right]^{1/2}}$$

NOW THAT COHERENCE HAS BEEN DEFINED, IT IS APPROPRIATE TO DISCUSS ITS ESTIMATION. FROM EACH OF TWO FINITE DURATION MEMBER FUNCTIONS OF CAPITAL N SEGMENTS, WE WEIGHT EACH SEGMENT BY A SMOOTH WEIGHTING FUNCTION, COMPUTE ITS DISCRETE FOURIER TRANSFORM VIA AN FFT, AND DENOTE THEM A SUB n AND B SUB n. AT ANY PARTICULAR FREQUENCY, THE COMPLEX COHERENCE IS ESTIMATED BY COMPUTING THE THREE SUMMATIONS SHOWN OVER THE AVAILABLE CAPITAL N SEGMENTS. THE LOWER CASE n DENOTES THE n-TH DATA SEGMENT AND THE FREQUENCY INDICATOR IS NOT SHOWN. IN THE NUMERATOR, WE MULTIPLY THE FFT OF THE A PROCESS BY THE COMPLEX CONJUGATE OF THE FFT OF THE B PROCESS AND SUM OVER N SEGMENTS TO OBTAIN AN ESTIMATE OF THE COMPLEX CROSS SPECTRUM. IN THE DENOMINATOR WE SUM THE MAGNITUDE SQUARED FFTS OVER THE N TIME SEGMENTS. UNDER CERTAIN SIMPLIFYING ASSUMPTIONS GIVEN IN THE CONFERENCE RECORD WE CAN DETERMINE THE STATISTICS OF THIS ESTIMATOR.

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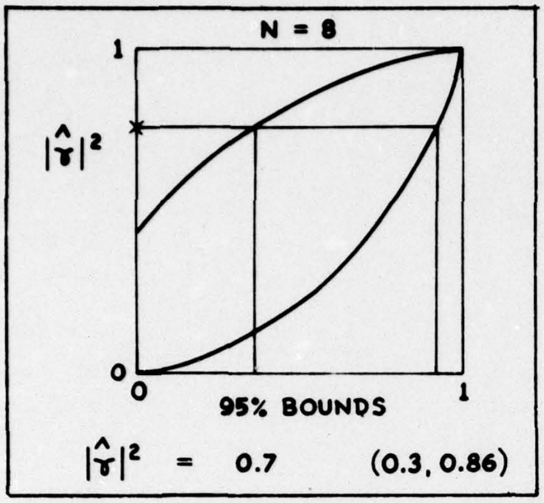

CONCLUSIONS

- **COHERENCE**
  - **NORMALIZED CROSS SPECTRUM**
  - **SIGNAL TO NOISE MEASURE**
  - **LINEARITY MEASURE**
- **ESTIMATION**
  - **DIFFICULT**
  - **BOUNDS LARGE**

In the conference record we discuss how to determine the confidence bounds. For a particular number of FFT averages (N = 8) and a prespecified confidence bound (95%), we obtain the two curves sketched here. When we obtain an estimate of gamma squared from the same number of FFTs as used to draw the curves, we use these curves to determine confidence bounds. In particular, if we have an estimate denoted by an X on the ordinate, we draw a horizontal line from the X until it intersects both curves. Then we drop two vertical lines to the abscissa and these are the confidence bounds. We can then state that the true value of gamma squared lies in the region bounded by the two abscissa values with the prespecified confidence. For example, with eight FFTs and an estimate of 0.7, the 95% confidence bounds are 0.3 and 0.86. With 128 FFTs and an estimate of 0.3, the bounds are 0.2 and 0.38. Thus, the bounds are large even when the number of FFTs is large.

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In conclusion, we have looked at what the coherence is. We have seen that it is a normalized cross spectrum that can provide a measure of signal-to-noise ratio and the extent to which the ocean medium can be modeled by a linear filter. In terms of measuring coherence, we have presented estimation equations that depend on the application of smooth weighting functions and large numbers of. of FFTs. These computational difficulties result in large bounds on the coherence estimates.

In summary, the coherence is an extremely useful descriptor in underwater acoustics that can be estimated with careful attention to detail and large numbers of FFTs.

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Are there any questions?

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CONFIDENCE BOUNDS FOR MAGNITUDE-SQUARED COHERENCE ESTIMATES

by

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ABSTRACT

In underwater acoustics where signals are digitally processed at the outputs of two or more receiving sensors, it is desirable to estimate the coherence spectrum, both for detection and position estimation. A processing technique for computing arbitrary confidence bounds for stationary Gaussian signals is presented. New computationally difficult examples are given for 80 to 95% confidence with independent averages of 8, 16, 32, 64 and 128. A discussion of the computational difficulties together with algorithmic details are presented.

INTRODUCTION

The magnitude-squared coherence (MSC) between two jointly stationary random processes x(t) and y(t) is defined as

$$C_{xy}(f) = \frac{|G_{xy}(f)|^2}{G_{xx}(f)G_{yy}(f)}, \qquad (1)$$

where $G_{xy}(f)$ is the cross-spectral density at frequency f and $G_{xx}(f)$ and $G_{yy}(f)$ are the autospectral densities. The MSC can be estimated as in [1] by

$$\hat{C}_{xy}(f) = \frac{\left|\sum_{n=1}^{N} X_n(f)Y_n^*(f)\right|^2}{\sum_{n=1}^{N}|X_n(f)|^2\sum_{n=1}^{N}|Y_n(f)|^2} \qquad (2)$$

where * denotes complex conjugate, N is the number of data segments employed, and $X_n(f)$ and $Y_n(f)$ are the Fast Fourier Transform (FFT) outputs of the nth data segments of x(t) and y(t). Both the MSC and its estimates are bounded by zero and unity. The cummulative distribution functions (CDF) for the MSC estimate in (2) have been determined in [1] under the assumptions that 1) the data are jointly stationary Gaussian random processes; 2) the N data segments are independent; 3) the data segments have been multiplied by a smooth weighting function to reduce sidelobe leakage; and 4) each data segment is sufficiently long to ensure adequate spectral resolution.

The MSC is useful in detection, see for example [2] and [3], but is also of value in estimating the amount of coherent power common between two received signals. Therefore it would be desirable having estimated a particular value of MSC to state with certain confidence that the true coherence falls in a specified interval. Early attempts to do this for 95% confidence were accomplished by Haubrich [4] who apparently used precomputed CDF curves and used a different method of presentation than the one used here. Related confidence work for the magnitude coherence (MC) or squareroot of (2) is presented by Koopmans [5]. Empirical results for 95% confidence are given by Benignus [6].

DETERMINING CONFIDENCE BOUNDS

Let C be the true but unknown parameter and $\hat{C}$ be its estimate. Then there exists a family of CDFs such as the two sketched in Fig. (1) for all values of C and N. For a fixed value of N, a number of

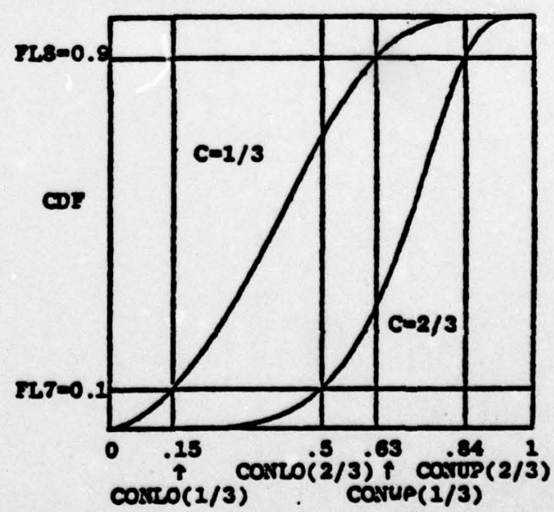

FIG. (1). PLOT OF CDF CURVES FOR N=8, C=1/3, AND N=8, C=2/3

CDF curves, such as plotted in Fig. (1), are generated, for various values of C. For each of the numerous CDF curves, we select, as closely as possible, the abscissa values such that the ordinate values FL8 minus FL7 yield the desired confidence. The confidence intervals are not unique, since there is no constraint such as FL8 equal FL7. We have selected FL8 equal FL7 but could have selected FL8 and FL7 such that the difference in abscissa values in Fig. (1) CONUP(C) minus CONLO(C) was minimum. However, as long as FL8 minus FL7 equals the desired confidence the method presented here is correct. Now we plot CONUP(C) and CONLO(C) versus C for this particular value of N. A result is sketched in Fig. (2).

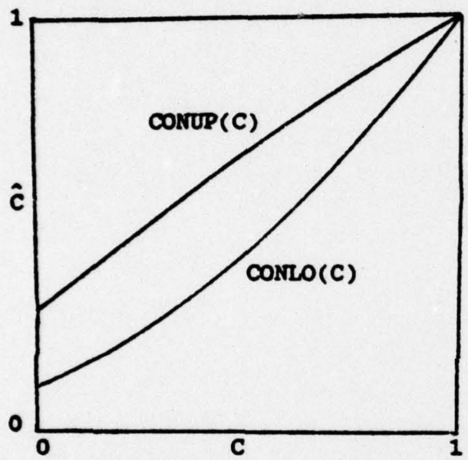

FIG. (2). HANDSKETCH OF CONFIDENCE BOUNDS FOR A PARTICULAR VALUE OF N

MAKING CONFIDENCE STATEMENTS ABOUT MSC ESTIMATES

A computer program has been written to evaluate the CDF and confidence limits. The mathematical details of the CDF as a finite sum of F21 hypergeometric functions, each one a polynomial, are given in [2]. For large values of N and C, a brute force approach to computing the CDF results in numeric overflows, attempts to avoid this problem can result in underflows or other inaccuracies. The program listed in the Appendix avoids these difficulties, it also incorporates CDF values when C equals zero or unity, since these can be computed in closed form.

Figures (3a) and (3b) are computer generated 80% and 95% confidence limits, respectively. The five pairs of curves in each figure are for N = 8, 16, 32, 64, and 128 from outer to inner, respectively. Having made an estimate with a particular value of N, only one pair of curves applies. An excellent discussion of the types of statements that can be made with confidence bounds is given by Cramér [7]. Suppose we obtain an estimated MSC of 0.7 from N = 8 disjoint FFTs, then we draw a horizontal line from 0.7 on Fig. (3b) for 95% confidence limits and see where it intersects the pair of N = 8 (outer) curves. This occurs at (approximate abscissa values) 0.3 and 0.86. Thus we state with 95% confidence that the true but unknown parameter C falls in the interval (0.3, 0.86). No matter what the true value of C, we have a 5% probability of giving an incorrect statement. That is, if we make many estimates of MSC and keep applying the rule described (whether or not C is random or constant) we will correctly include the true value of C in the interval that we specify 95% of the time. Sometimes the method of applying the rule is in doubt as for example in Fig. (3b) if the estimate comes out to be 0.3 and N = 8 then a horizontal line does not intersect the upper confidence limit curve unless we extrapolate it backwards. Doing this means making statements like: with 95% confidence the true MSC is in the region (-0.1, 0.62). Since we know apriori that the true value of C is non-negative, we could just as easily say (but with no more confidence) that with 95% confidence (for N = 8 and $\hat{C}$ = 0.3) the true MSC falls in the region (0.0, 0.62). Moreover, if both intersections result in negative regions (as for example when $\hat{C}$ = 0.001 and N = 8) we may have to make statements like with 80% confidence the true MSC lies in (0.0, 0.0). However, if we continue to apply the rule and run the experimental trials we will make correction statements "80%" of the time. It is interesting to note that due to the properties of the estimate and our selection of FL7 and FL8 that larger values of N do not always result in the upper confidence bound being lower. This also occurs in MC estimate confidence limits [5]. It is also interesting to note that while increasing N is desirable, the confidence bounds for N = 128 are still very large. For example, even when N = 128 if $\hat{C}$ = 0.3 the 95% confidence intervals are still (0.2, 0.38) and the 80% confidence intervals (0.24, 0.36) are not much better.

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SAMPLE OUTPUT FROM PROGRAM LISTED IN APPENDIX

8	.000	.015	.280	.004	.410
8	.167	.055	.487	.015	.611
8	.333	.154	.626	.061	.7[illegible]7
8	.500	.305	.741	.175	.815
8	.667	.500	.838	.365	.886
8	.833	.732	.924	.637	.947
8	1.000	1.000	1.000	1.000	1.000
16	.000	.007	.142	.002	.218
16	.167	.065	.372	.023	.466
16	.333	.190	.533	.111	.614
16	.500	.354	.669	.259	.732
16	.667	.546	.789	.458	.832
16	.833	.762	.899	.705	.921
16	1.000	1.000	1.000	1.000	1.000
32	.000	.003	.072	.001	.112
32	.167	.084	.307	.044	.369
32	.333	.225	.470	.164	.532
32	.500	.393	.618	.327	.663
32	.667	.580	.754	.523	.789
32	.833	.783	.882	.747	.899
32	1.000	1.000	1.000	1.000	1.000
64	.000	.002	.036	.000	.057
64	.167	.102	.258	.071	.308
64	.333	.253	.426	.209	.478
64	.500	.423	.586	.377	.625
64	.667	.605	.729	.567	.758
64	.833	.793	.869	.775	.889
64	1.000	1.000	1.000	1.000	1.000
128	.000	.001	.018	.000	.029
128	.167	.119	.230	.094	.269
128	.333	.275	.407	.243	.447
128	.500	.445	.565	.413	.599
128	.667	.623	.718	.597	.740
128	.833	.808	.880	.793	.879
128	1.000	1.000	1.000	1.000	1.000

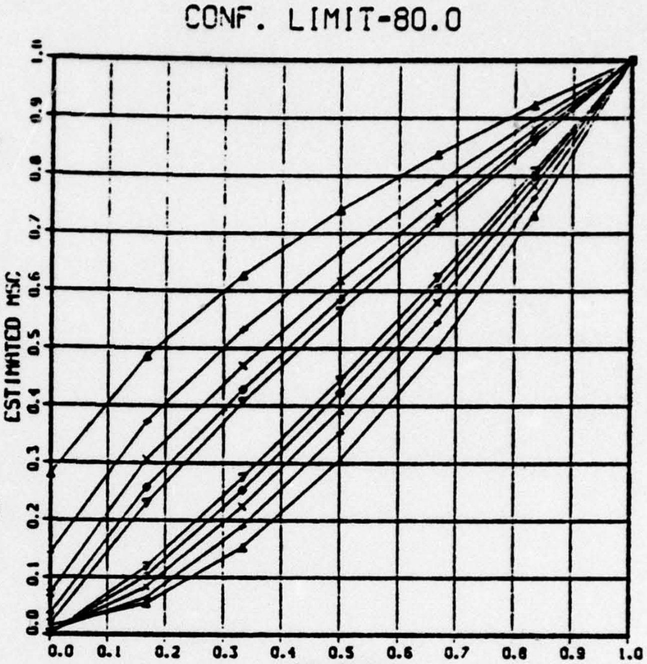

FIG. (3a). 80% CONFIDENCE LIMITS FOR N = 8, 16, 32, 64, AND 128

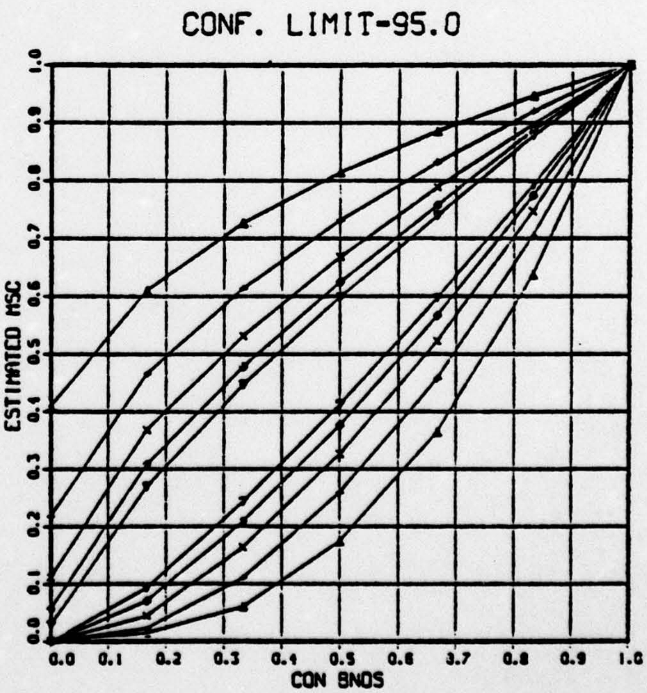

FIG. (3b). 95% CONFIDENCE LIMITS FOR N = 8, 16, 32, 64, AND 128

APPENDIX. PROGRAM FOR CONFIDENCE BOUNDS

```
      PARAMETER NC=7,NN=7
      DIMENSION B(101),LAB(5),FIL(2)
      DOUBLE PRECISION C2,C3,C4,T,E,T2,F
      DIMENSION CONUP(2,NN,NC),CONLO(2,NN,NC)
      DIMENSION X(NC),Y1(NC),Y2(NC)
      DO 50 IL=1,2
      DO 50 IJ=1,NN
      DO 50 IC=1,NC
      CONUP(IL,IJ,IC)=1.0
      CONLO(IL,IJ,IC)=0.0
   50 CONTINUE
      LOOP=NC-2
  100 DO 600 IC=1,LOOP
      C=FLOAT(IC)/FLOAT(NC-1)
  200 DO 590 IL=1,2
      FL6=65+(IL*15)
  300 DO 580 IJ=3,NN
      N=2**IJ
  400 DO 410 I=1,101
      B(I)=1.0
  410 CONTINUE
      A=1.0-FLOAT(N)
      FL7=(100.0-FL6)/200.0
      FL8=1.0-FL7
      TEMP=1.0/FLOAT(N-1)
      CONLO(IL,IJ,1)=1.0-(FL8**TEMP)
      CONUP(IL,IJ,1)=1.0-(FL7**TEMP)
      CONUP(IL,IJ,NC)=1.0
      CONLO(IL,IJ,NC)=1.0
      DO 510 K=1,100
      E= FLOAT(K-1)/100.0
      Z=E*C
      P=0
      IF (Z .EQ. 0) GO TO 480
      C4=(1-E)/(1-Z)
      T2=(1-C)/(1-Z)
      C2=E*T2**N
      IF(E .EQ. 0)  GO TO 480
      IR=N-2
      DO 470 L=0,IR
      C3=C4**L
      T=C3
      F=T
      IF(L.EQ.0)GO TO 455
      DO 450 K2=1,L
      K1=K2-1
      FK=FLOAT(K2)
      T=T*Z*(FLOAT(A+K1)/FK)*(FLOAT(K1-L)/FK)
      IF(T.LT.0.0001*F*L)GO TO 455
      F=F+T
  450 CONTINUE
  455 CONTINUE
      P=P+C2*F
      IF(P.GT.FL8)GO TO 480
  470 CONTINUE
  480 B(K)=P
      IF(P.GT.FL8)GO TO 515
  510 CONTINUE
  515 CONTINUE
      DO 570 I=1,100
      IF (B(I).GE.FL7) GO TO 520
      IF (B(I+1).LT.FL7) GO TO 520
      D2=B(I+1)-B(I)
      E=((I-1)/100.0) + ((FL7-B(I))/(100*D2))
      CONLO(IL,IJ,IC+1)=E
  520 IF (B(I).GE.FL8) GO TO 570
      IF (B(I+1).LT.FL8) GO TO 570
      D3 =B(I+1) -B(I)
      E=((I-1)/100.0) + ((FL8-B(I))/(100*D3))
      CONUP(IL,IJ,IC+1)=E
  570 CONTINUE
  580 CONTINUE
  590 CONTINUE
  600 CONTINUE
      DO 640 IL=1,2
      DO 640 IJ=3,NN
      N=2**IJ
      DO 640 IC=1,NC
      COH=(IC-1)/FLOAT(NC-1)
      F1=CONLO(1,IJ,IC)
      F2=CONUP(1,IJ,IC)
      F3=CONLO(2,IJ,IC)
      F4=CONUP(2,IJ,IC)
      PRINT 630 N,COH,F1,F2,F3,F4
  630 FORMAT(1X,I5,5F6.3)
  640 CONTINUE
      CALL COMPRS
      DO 750 IL=1,2
      FIL=65+(IL*15)
      ENCODE(30,651,LAB)FIL
  651 FORMAT('CONF. LIMIT=',F4.1,' %S')
      CALL TITLE(LAB,100,'CON BNDSS',100,
     'ESTIMATED MSCS',100,6.,6.)
      CALL FRAME
      CALL GRAF(0.0,0.1,1.0,0.0,0.1,1.0)
      CALL GRID(1,1)
      DO 740 IJ=3,NN
      DO 700 IC=1,NC
      X(IC)=FLOAT(IC-1)/FLOAT(NC-1)
      Y1(IC)=CONUP(IL,IJ,IC)
      Y2(IC)=CONLO(IL,IJ,IC)
  700 CONTINUE
      ITEMP=IJ-1
      CALL MARKER(ITEMP)
      CALL CURVE(X,Y1,NC,1)
      CALL CURVE(X,Y2,NC,1)
  740 CONTINUE
      CALL ENDPL(IL)
  750 CONTINUE
      CALL DONEPL
      END
```

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